# Rigorous Dimension Estimates for Fractals and How to Find Them

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# $t$-Hausdorff Measure

## Definition ($t$-Hausdorff Measure)

*For any* $t \in [0, \infty)$, *let*

$$\mathcal{H}^t_\delta(S) = \inf_{\substack{\mathcal{U} \text{ is an open} \\ \text{cover of } S}} \left\{ \sum_{i=1}^{\infty} \textit{diam}(U_i)^t : U_i \in \mathcal{U},\ \textit{diam}(U_i) < \delta \right\}.$$

*The* $t-$*dimensional Hausdorff measure is given by*

$$\mathcal{H}^t(S) = \lim_{\delta \to 0} \mathcal{H}^t_\delta(S).$$

# $t$-Hausdorff Measure Visualized

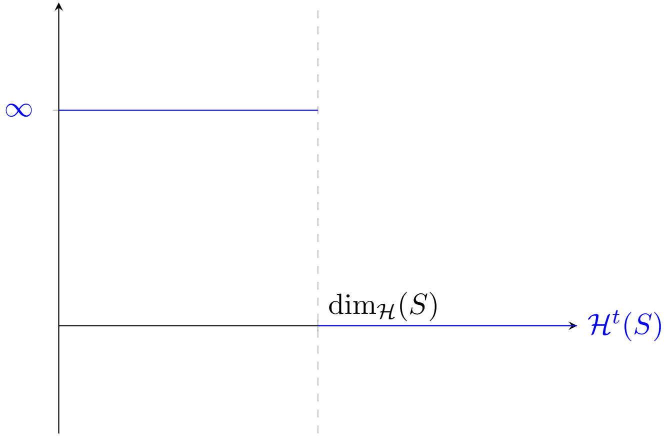


Figure: An example plot of $\mathcal{H}^t(S)$ as a function of $t$.

# Hausdorff Dimension

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$$\dim_{\mathcal{H}}(S) = \inf\{t \geq 0 : \mathcal{H}^t(S) = 0\}.$$

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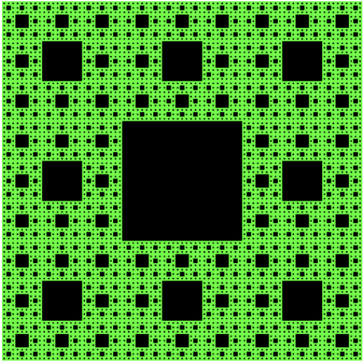
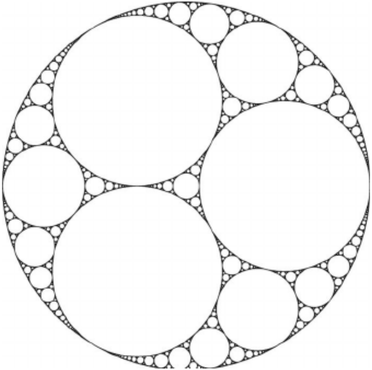
- Hausdorff dimension agrees with standard notions of dimension.
- Hausdorff dimension is a *bi-Lipschitz* equivalence.

# Two Famous Fractals



(a) The *Sierpinski carpet*.



(b) The *Apollonian gasket*.

Both of these are the limit set of *iterated function systems*.

# Iterated Function Systems

## Definition (Iterated Function System)

*An Iterated Function System (IFS)* $\mathcal{S} = \{X, E, \{\phi_e\}_{e \in E}\}$ *consists of:*

1. *a compact metric space* $X$*,*
2. *a countable set* $E$ *with at least 2 elements, and*
3. *a family of injective contractions* $\{\phi_e : X \to X\}_{e \in E}$ *with uniform Lipschitz constant* $s \in (0, 1)$*.*

There is a unique compact set $K \subset X$ so that

$$K = \cup_{e \in E} \phi_e(K).$$

This is called the *limit set* of the IFS, often denoted $J_E$.

# Hutchinson's Theorem

## Theorem (Hutchinson's Theorem)

*For an IFS $\mathcal{S}$ consisting of metric similarities, then*

$$h_{\mathcal{S}} = \dim_{\mathcal{H}}(J_{\mathcal{S}}) = \inf \left\{ t \geq 0 : \sum_{e \in E} \|D\phi_e\|_\infty^t < 1 \right\}.$$

Due to Hutchinson (1981).

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Due to Hutchinson (1981). This theorem can readily give the dimension of the Sierpinski carpet, but does not apply for the Apollonian gasket.

# An Application of Hutchinson's Theorem

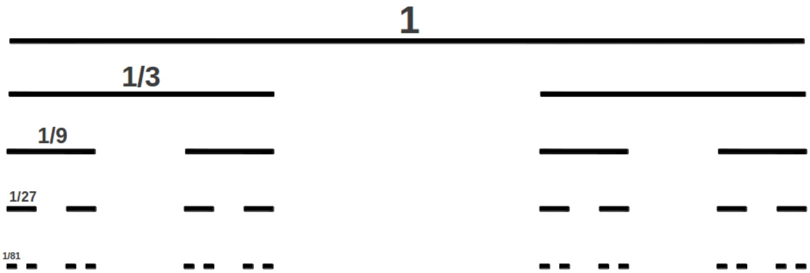


Figure: The Cantor set generated by $\phi_1(x) = \frac{x}{3}$, $\phi_2(x) = \frac{2}{3} + \frac{x}{3}$.

# An Application of Hutchinson's Theorem

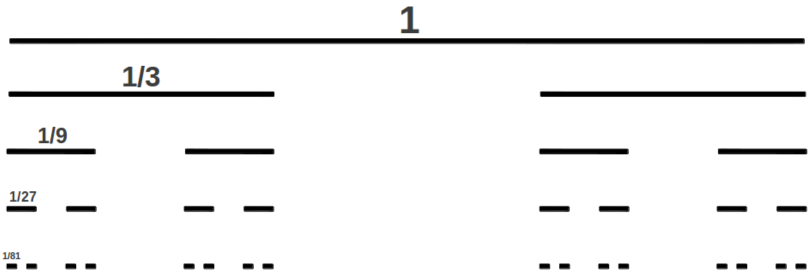


Figure: The Cantor set generated by $\phi_1(x) = \frac{x}{3}$, $\phi_2(x) = \frac{2}{3} + \frac{x}{3}$.

$$\|D\phi_1\|_\infty^t + \|D\phi_2\|_\infty^t = \frac{1}{3}^t + \frac{1}{3}^t = 2\left(\frac{1}{3}\right)^t, \text{ so } 2\left(\frac{1}{3}\right)^t = 1 \Rightarrow t = \frac{\log 2}{\log 3}.$$

# Applications of Conformal Dimension Estimates

Dimension estimates for conformal fractals are used in the following areas:

- Zaremba theory (Bourgain and Kontorovich (2014))
- Patterson Sullivan Theory
- Scattering theory on hyperbolic 3-manifolds (Borthwick, McRae, and Taylor (1997))
- Markov and Lagrange Spectra

# Linear Conformal Maps

## Definition (Linear Conformal Map)

*A nonsingular linear map* $T : \mathbb{R}^n \to \mathbb{R}^n$ *is conformal if one of the following (equivalent) conditions holds for all* $x, y \in \mathbb{R}^n \setminus \{0\}$ *:*

1. $\angle(x, y) = \angle(T(x), T(y))$.
2. *There exists some* $\lambda > 0$ *such that* $\langle T(x), T(y)\rangle = \lambda\langle x, y\rangle$.
3. *There exists some* $\mu > 0$ *so that* $|Tx| = \mu|x|$.

# Conformal Diffeomorphisms

**Definition (Conformal Diffeomorphism)**

*For an open set* $U \subset \mathbb{R}^n$, *a* $C^1$ *diffeomorphism is called conformal at* $x \in U$ *if it's derivative* $Df(x) : \mathbb{R}^n \to \mathbb{R}^n$ *is a linear conformal map.*

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*For an open set $U \subset \mathbb{R}^n$, a $C^1$ diffeomorphism is called conformal at $x \in U$ if it's derivative $Df(x) : \mathbb{R}^n \to \mathbb{R}^n$ is a linear conformal map.*

For an open set $U \subset \mathbb{R}^n$, if $f : U \to \mathbb{R}^n$ is conformal then

1. if $n = 1$, $f$ is a monotone $C^1$ diffeomorphism,
2. if $n = 2$, then $f$ is either holomorphic or antiholomorphic,
3. if $n \geq 3$, then $f$ is a Möbius map (Liouville's Theorem).

# Conformal Iterated Function Systems

An IFS $\{X, E, \{\phi_e\}_{e \in E}\}$ is called a *conformal iterated function system* (CIFS) if the following hold:

1. $\overline{\text{Int}(X)} = X$.
2. (*Open Set Condition*) For all distinct $a, b \in E$,

$$\phi_a(\text{Int}(X)) \cap \phi_b(\text{Int}(X)) = \emptyset.$$

3. There exists an open connected set $W \supset X$ such that for all $e \in E$, the maps $\phi_e$ extend to conformal maps taking $W$ into $W$.

# Conformal Iterated Function Systems cont.

4. (*Bounded Distortion Property*) There exists a compact and connected set $S$ so that

$$X \subset \text{Int}S \subset S \subset W$$

and two constants $L \geq 1$ and $\alpha > 0$ such that

$$\left| \frac{\|D\phi_e(x)\|}{\|D\phi_e(y)\|} - 1 \right| \leq L|x-y|^\alpha$$

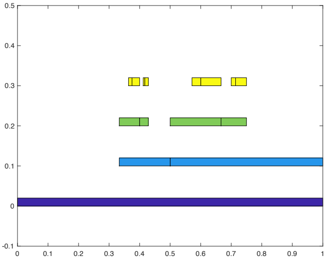
for every $e \in E$ and each pair of points $x, y \in S$, where

$$\|D\phi_e(x)\| = \sup\{|D\phi_e(x)(p)| : p \in \mathbb{R}^n,\ |p| \leq 1\}.$$

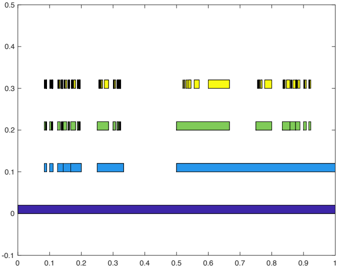
# The Continued Fraction CIFS

The system $\mathcal{CF}_E = \{X, E, \{\phi_e\}_{e \in E}\}$ where $X = [0,1]$, $E \subseteq \mathbb{N}$ and

$$\phi_e(x) = \frac{1}{x+e}.$$

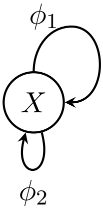



(a) $E = \{1, 2\}$.




(b) $E = \{1, 3, 4, 5, 7, 9, 11\}$.

# Graphical Representation




An example IFS consisting of two maps.

# Graph Directed Markov Systems

A *graph directed Markov system* (GDMS)

$$\mathcal{S} = \{V, E, A, t, i, \{X_v\}_{v\in V}, \{\phi_e\}_{e\in E}\} \tag{1}$$

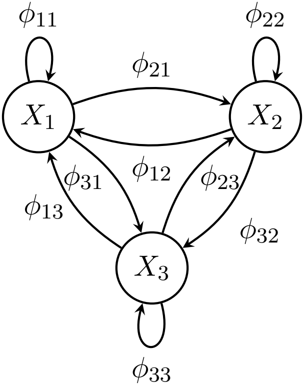
as defined in Mauldin-Urbański (2003) consists of

1. a directed multigraph $(E, V)$ with a countable set of edges $E$, called the *alphabet* of $\mathcal{S}$, and a finite set of vertices $V$,
2. an incidence matrix $A : E \times E \to \{0, 1\}$,
3. functions $i, t : E \to V$ so that $t(a) = i(b)$ whenever $A_{ab} = 1$,
4. a family of non-empty compact metric spaces $\{X_v\}_{v\in V}$,
5. a family of injective contractions

   $$\left\{\phi_e : X_{t(e)} \to X_{i(e)}\right\}_{e\in E}$$

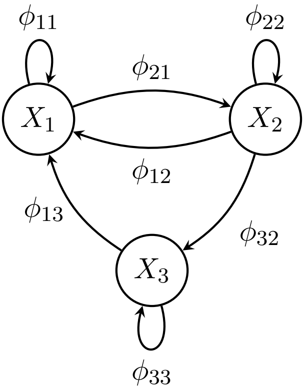
   with uniform Lipschitz constant $s \in (0, 1)$.

# GDMS Example




An example *graph directed Markov system* (GDMS).

# GDMS with Removed Edges




Removing edges from a GDMS.

# CGDMS

A GDMS is called a *conformal graph directed Markov system* if it consists of conformal diffeomorphisms.

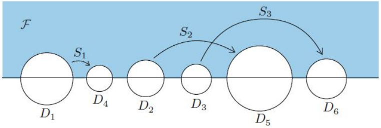



Figure: Schottky groups are natural examples of CGDMSs.

# Symbolic Dynamics pt 1

For $n \in \mathbb{N}$, let

$$E_A^n = \left\{\omega \in E^n : A_{\omega_i \omega_{i+1}} = 1,\ i \in \{1, 2, \dots, n-1\}\right\}.$$

$E_A^n$ is the set of *words* of length $n$. The set of *words of finite length* is $E_A^*$, and the set of all *infinite words* is $E_A^{\mathbb{N}}$.
Given $\tau \in E_A^n$, the map coded by $\tau$ is given by

$$\phi_\tau = \phi_{\tau_1} \circ \phi_{\tau_2} \circ \cdots \circ \phi_{\tau_n} : X \to X, \text{ where } \tau \in E_A^n.$$

# Symbolic Dynamics pt 2

For any $\omega \in E_A^{\mathbb{N}}$, $(\phi_{\omega|_n}(X))_{n=1}^{\infty}$ is a descending sequence of compact sets. By *Cantor's Intersection Theorem*

$$\bigcap_{n \in \mathbb{N}} \phi_{\omega|_n}(X)$$

is singleton. Define the coding map by

$$\pi : E_A^{\mathbb{N}} \to X, \;\; \pi : \omega \mapsto \bigcap_{n \in \mathbb{N}} \phi_{\omega|_n}(X).$$

The limit set of $\mathcal{S}$ is $J_E = \pi(E_A^{\mathbb{N}})$.

# Topological Pressure

Given some $n \in \mathbb{N}$, $t \geq 0$, the $n$*-th partition function* is defined as

$$Z_n(t) = \sum_{\omega \in E_A^n} \|D\phi_\omega\|_\infty^t.$$

As a sequence, $(Z_n(t))_{n=1}^\infty$ is submultiplicative, implying that $(\log Z_n(t))_{n=1}^\infty$ is subadditive. The *topological pressure* is defined as

$$P(t) = \lim_{n\to\infty} \frac{\log(Z_n(t))}{n} = \inf_{n\in\mathbb{N}} \frac{\log(Z_n(t))}{n}.$$

# Dimension Theory: Definitions

A nonnegative number $t$ belongs to Fin$(\mathcal{S})$ if

$$Z_1(t) = \sum_{e \in E} \|D\phi_e\|_\infty^t < \infty.$$

The number

$$h_{\mathcal{S}} := \inf \{t \geq 0 : P(t) \leq 0\}$$

is called *Bowen's parameter*. The $\theta$-*number* for $\mathcal{S}$ is given by

$$\theta := \theta_{\mathcal{S}} = \inf \text{Fin}(\mathcal{S}).$$

# Properties of the Pressure Function

For a CGDMS $\mathcal{S}$, the following conclusions hold:

1. $\text{Fin}(\mathcal{S}) = \{t \geq 0 : P(t) < \infty\}$.
2. $\theta = \inf\{t \geq 0 : P(t) < \infty\}$.
3. The topological pressure $P$ is strictly decreasing on $[\theta, \infty)$ with $P(t) \to -\infty$ as $t \to \infty$. Moreover, $P$ is convex on the closure of $\text{Fin}(\mathcal{S})$.
4. $P(0) = \infty$ if and only if $E$ is infinite.

# Pressure Visualized

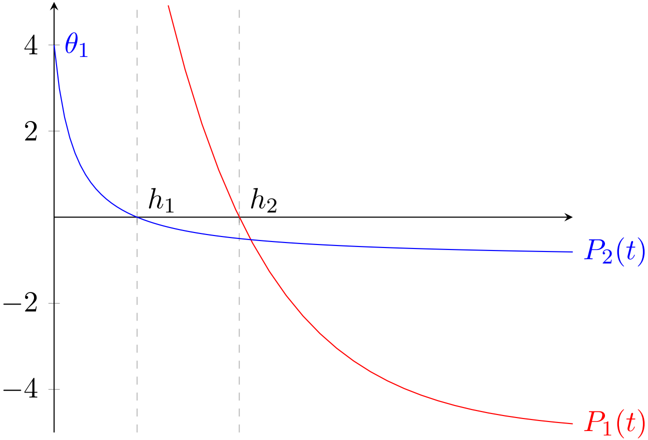



Figure: Two example pressure functions $P_1(t)$ and $P_2(t)$.

# Bowen's Formula

**Theorem (Bowen's Formula)**

*If $\mathcal{S}$ is a CGDMS, then*

$$h_{\mathcal{S}} := \dim_{\mathcal{H}}(J_{\mathcal{S}}) = \sup\left\{\dim_{\mathcal{H}}(J_F) : F \subset E \text{ is finite}\right\}.$$

Due to Mauldin and Urbański: (1996) for CIFSs, (2003) for CGDMSs.

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Due to Mauldin and Urbański: (1996) for CIFSs, (2003) for CGDMSs.
This is a key tool in rigorous dimension estimates, but computing $P(t)$ directly is unfeasible.

# The Symbolic Transfer Operator

Suppose $\mathcal{S}$ is a conformal GDMS (CGDMS) and $t \in \text{Fin}(\mathcal{S})$.

## Definition (Symbolic Transfer Operator)

*For $g \in C_b(E_A^{\mathbb{N}})$ and $\omega \in E_A^{\mathbb{N}}$, the symbolic transfer operator is given by $\mathcal{L}_t : C_b(E_A^{\mathbb{N}}) \to C_b(E_A^{\mathbb{N}})$*

$$\mathcal{L}_t g(\omega) = \sum_{i:A_{i\omega_1}=1} g(i\omega)\|D\phi_i(\pi(\omega)\|^t.$$

# Spectral Properties

The following theorem was proven in Mauldin-Urbański (2003)

## Theorem

*The spectral radius of $\mathcal{L}_t$ is $e^{P(t)}$. This eigenvalue is isolated, corresponds to the unique, positive eigenfunction $\rho_t$ of $\mathcal{L}_t$, and*

$$\rho_t = \frac{d\mu_t}{dm_t},$$

*where $\mu_t$ is the pushforward of the unique shift-invariant Gibb's measure and $m_t$ is the $t$-conformal measure.*

This theorem allows for the use of *Rayleigh quotients* to calculate $\lambda_t := e^{P(t)}$.

# The Spatial Transfer Operator

## Theorem (Chousionis, Leykekhman, Urbański, W)

*Suppose that $\mathcal{S}$ is a finitely irreducible, maximal CGDMS, and let $t \in [0, \infty)$ be so that $P(t) < \infty$. Then:*

1. *There exists a unique continuous function $\rho_t : X \to [0, \infty)$ so that*
$$F_t \rho_t = \rho_t \text{ and } \int \rho_t dm_t = 1.$$
2. $K^{-t} \leq \rho_t \leq K^t / M_t$ *where* $M_t = \min\{m_t(X_v) : v \in v\}$.
3. *The sequence $\{F_t^n(1)\}_{n=1}^{\infty}$ converges uniformly to $\rho_t$ on $X$.*
4. $\rho_t|_{J_{\mathcal{S}}} = \frac{d\mu_t}{dm_t}$.
5. $\rho_t$ *can be extended to a real analytic function in an open neighborhood of $X$.*

# Spectral Properties 2

The spatial transfer operator has the same spectral properties as the symbolic one. That is,

- The leading eigenvalue $\lambda_t$ of $F_t$ is $e^{P(t)}$.
- $\lambda_t$ corresponds to the unique positive eigenfunction of $F_t$.
- There is a spectral gap.

# Discretizing $C(X)$

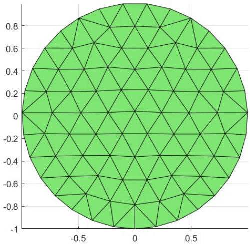
Suppose that $X^h$ is a *conformal mesh* of $X$ with nodes $\{x_j\}_{j=1}^N$. For $\tau \in X^h$ let $h_\tau = \mathrm{diam}(\tau)$ and define $h = \max_{\tau \in X^h} h_\tau$. Consider the space

$$S_h := \{v \in C(X) : v \in \mathcal{P}_1(\tau) \text{ for all } \tau \in X^h\}$$

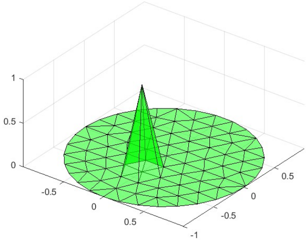
with nodal basis $\{\phi_j\}_{j=1}^N$, and the *interpolation operator*

$$\mathcal{I}_h(v)(x) = \sum_{j=1}^N v(x_j)\phi_j(x), \quad \mathcal{I}_h : C(X) \to S_h.$$

# A Mesh and a Basis Function




(a) An example mesh of $\mathbb{D}$.




(b) A shape function on the mesh.

# Bramble-Hilbert

Set $\rho_t^I = \mathcal{I}_h \rho_t$. Then, as an application of the *Bramble-Hilbert* Lemma,

$$\left\| \rho_t - \rho_t^I \right\|_\infty \leq C_{BH} h^2 \left\| D^2 \rho_t \right\|_\infty$$

for a computable constant $C_{BH}$. A bound on $\left\| D^2 \rho_t \right\|_\infty$, uniform in $h$, is enough for convergence of the method. Our method uses a bound of the form

$$\left\| D^2 \rho_t \right\|_\infty \leq C \| \rho_t \|_\infty,$$

to imply $\left\| \rho_t - \rho_t^I \right\|_\infty$ will decrease on the order of $h^2$.

# Eigenfunction Estimates

## Theorem (Chousionis, Leykekhman, Urbański, W)

*Let $\mathcal{S} =$ be a a finitely irreducible, maximal CGDMS. If $t \in$ Fin$(\mathcal{S})$, and $\alpha$ is any multiindex, then there are computable constants $C(\alpha, t, n)$ and $C_2(\alpha, t)$ so that*

1. *if $n \geq 3$,*

$$|D^\alpha \rho_t(x)| \leq C(\alpha, t, n)\rho_t(y), \quad \forall x, y \in X_v, v \in V, \text{ and} \tag{2}$$

2. *if $n = 2$, then*

$$|D^\alpha \rho_t(x)| \leq C_2(\alpha, t)\rho_t(x), \quad \forall x \in X. \tag{3}$$

# Bounding $\left\|\rho_t - \rho_t^I\right\|_\infty$

For any $\tau \in X^h$ and any $x \in \tau$, set

$$err_\tau = C_{BH}(c_1 h_\tau + 1)c_2 h_\tau^2$$

where $c_1$ and $c_2$ correspond to the previous derivative bounds with $\alpha = 1, 2$. Then,

$$(1 - err_\tau)\rho_t^I(x) \leq \rho_t(x) \leq (1 + err_\tau)\rho_t^I(x)$$

and

$$(1 - err_\tau)F_t\rho_t^I(x) \leq F_t\rho_t(x) \leq (1 + err_\tau)F_t\rho_t^I(x).$$

# Finding $\rho_t^I$

To find $\rho_t^I$ we assemble discrete versions of $F_t$. Define two matrices $A_t, B_t \in \mathbb{R}^{N\times N}$ such that

$$(A_t\alpha)_j := (1-\text{err})\sum_{e\in E_A} \|D\phi_e(x_j)\|^t \mathcal{I}_h\rho_t(\phi_e(x_j))\chi_{X_{t(e)}}(x_j)$$
$$(B_t\alpha)_j := (1+\text{err})\sum_{e\in E_A} \|D\phi_e(x_j)\|^t \mathcal{I}_h\rho_t(\phi_e(x_j))\chi_{X_{t(e)}}(x_j).$$

Then $\rho_t^I = \mathcal{I}_h(\rho_t)$ may be found using *barycentric approximation* to assemble the appropriate approximation matrix and finding its leading eigenvector.

# Results for Positive Matrices

The following lemma was used by Falk and Nussbaum (2016), and is a key to our rigorous approximation results.

## Lemma

*Suppose that $M$ is a non-negative, $N \times N$ matrix. For a strictly positive vector $w \in \mathbb{R}^N$,*

$$\text{if } Mw \geq \lambda w, \text{ then } r(M) \geq \lambda$$

$$\text{if } Mw \leq \lambda w, \text{ then } r(M) \leq \lambda.$$

# Dimension Bounds

Combining all of these results, setting $\alpha_t$ to be the vector in $\mathbb{R}^N$ with entries $(\alpha_t)_j = \rho_t^I(x_j) = \rho_t(x_j)$, one has

$(A_t\alpha_t)_j \leq F_t\rho_t(x_j) = \lambda_t\rho_t(x_j)$ and $(B_t\alpha_t)_j \geq F_t\rho_t(x_j) = \lambda_t\rho_t(x_j)$.

Therefore,

$$r(A_t) \leq \lambda_t \leq r(B_t),$$

and bisection method may be applied to find rigorous bounds on Hausdorff dimension.

# Specific Dimension Estimates

Table: Hausdorff dimension estimates for various examples.

| Example | Hausdorff dimension |
|---|---|
| $2D$ Continued fractions with 4 generators | $1.149576 \pm 5.5e-06$ |
| $2D$ Continued fractions | $1.853 \pm 4.2e-03$ |
| $2D$ Continued fractions on Gaussian primes | $1.510 \pm 4.0e-03$ |
| $3D$ Continued fractions with 5 generators | $1.452 \pm 9.7e-03$ |
| $3D$ Continued fractions | $2.57 \pm 1.7e-02$ |
| A quadratic $abc$-example | $0.63271428571428 65 \pm 5.0e-16$ |
| An example of a Schottky group | $0.7753714285 \pm 1.5e-10$ |
| 12 map Apollonian subsystem | $1.0285714285713 \pm 1.1e-13$ |
| Apollonian gasket | $1.30565 \pm 5e-05$ |
| Apollonian gasket without a generator | $1.2196 \pm 2e-04$ |
| Apollonian gasket without a spiral | $1.2351 \pm 5.5e-04$ |

# A Fractal Meshing Algorithm

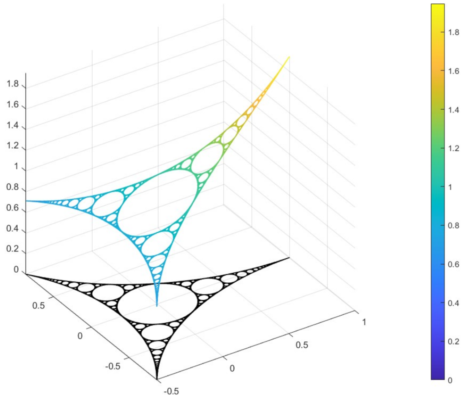



Figure: The eigenfunction approximation for the Apollonian gasket over a fractal mesh.

# Future Work

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- How can we generalize this to different spaces? Are there better bounds we can find thanks to Liouville's theorem?
- Can we apply this to solve the *Texan Conjecture* for different systems?

# Thanks for Coming

Thank you for coming!